Brigham Young University
Department of Electrical and
Computer Engineering

459 Clyde Building
Provo, Utah 84602

Electromagnetics in Two Dimensions

Karl F. Warnick

8 May 1995

MERS Technical Report # MERS 95-003
ECEN Department Report # TR-L100-95.3

Microwave Earth Remote Sensing (MERS) Research Group

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1. INTRODUCTION

In this paper, Maxwell's laws for the fields due to charged sources in a flat, two-dimensional world are determined from the assumptions that a Coloumb force between charged sources exists and that Maxwell's laws written using differential forms have the same form in two and three dimensions.

Maxwell's laws in three dimensions are

$$\begin{aligned} dE &= -\frac{\partial}{\partial t}B_m + J_m \\ dH &= \frac{\partial}{\partial t}D_m + J_e \\ dD &= \rho_e \\ dB &= \rho_m \\ D &= \star E \\ B &= \star H \end{aligned} \tag{1}$$

where E is the electric field intensity 1-form, B is the magnetic field intensity 1-form, D is the electric flux density 2-form, B is the magnetic flux density 2-form, J_m and J_e are the magnetic and electric current density 2-forms and ρ_m and ρ_e are the magnetic and electric charge density 3-forms. $\star$ is the Hodge star operator and d is the exterior derivative. We show below that by changing only the degrees of the forms in these equations, we can arrive at a self-consistent set of equations for electromagnetics in two dimensions.

Once this result is achieved, we treat wave propagation in two dimensions and show that Maxwell's laws in three dimensions for sources isotropic in one direction reduce to the same two-dimensional laws. Finally, we discuss Maxwell's laws in a curved two-dimensional space.

2. MAXWELL'S LAWS IN TWO-DIMENSIONAL SPACE FOR ELECTRIC SOURCES

We postulate the existence of electric charges that exert force on each other that goes as the product of the charges and inverse distance. We can define a 1-form E such that if a test charge q is displaced by an *infinitesimal* amount $\mathbf{x}$,

$$W = -q\mathbf{x} \rfloor E \tag{2}$$

where E is arbitrary and depends on the configuration of all other charges.

We can define electric flux D using the 2-dimensional star operator, which is defined for 1-forms by

$$\begin{aligned} \star dx &= dy \\ \star dy &= -dx. \end{aligned}$$

We make the definition $D = \star E$ (where units are normalized such that $\epsilon = \mu = 1$) so that D is a 1-form. Using the form of Gauss's law for electric flux density in three dimensions, we can write

$$dD = \rho \tag{3}$$

where ρ is a 2-form specifying the density of charges in 2-space.

By analogy with three dimensions, we write Faraday's law by setting the exterior derivative of the 1-form E equal to the time derivative of some form B,

$$dE = -\frac{\partial}{\partial t} B \tag{4}$$

which shows that B is a 2-form, representing a surface density in 2-space. Writing $B = \star H$ shows that H is a 0-form, or scalar. We can write Ampere's law,

$$dH = \frac{\partial}{\partial t} D + J \tag{5}$$

where J is a 1-form specifying the density of charge flow, $J = -\mathbf{v} \rfloor \rho$ where $\mathbf{v}$ is the velocity field of the charge density ρ. The final equation is trivial, $dB = 0$, since the exterior derivative of any 2-form in two dimensions is zero.

We now have all of Maxwell's laws, as well as the constitutive relationships:

$$dE_e = -\frac{\partial}{\partial t} B_e$$

$$
\begin{aligned}
dH_e &= \frac{\partial}{\partial t} D_e + J_e \\
dD_e &= \rho_e \\
dB_e &= 0 \\
D_e &= \star E_e \\
B_e &= \star H_e
\end{aligned}
\tag{6}
$$

where the subscript denotes electric sources and fields due to electric sources and units are normalized as described above. H is a 0-form, E_e, D_e and J_e are 1-forms and B_e and ρ_e are 2-forms,

3. MAXWELL'S LAWS FOR MAGNETIC SOURCES

If we postulate magnetic charge rather than electric charge, we find that B_m and H_m now become 1-forms, while E_m becomes a 0-form and D_m becomes a 2-form, where the subscript m indicates that these are fields due to magnetic charges, rather than electric charges as in the previous section. We arrive at an independent set of equations,

$$
\begin{aligned}
dE_m &= -\frac{\partial}{\partial t} B_m + J_m \\
dH_m &= \frac{\partial}{\partial t} D_m \\
dD_m &= 0 \\
dB_m &= \rho_m \\
D_m &= \star E_m \\
B_m &= \star H_m
\end{aligned}
$$

where ρ_m is a two-form giving the density of magnetic charge and J_m is the current density 1-form due to moving magnetic charge.

A simple proof that magnetic and electric charge cannot affect each other follows from inspection of the forms of the Lorentz force laws for the two systems:

$$F_{electric} = q_e(E + \mathbf{v} \lrcorner B) \tag{7}$$

$$F_{magnetic} = q_m(H + \mathbf{v} \lrcorner D) \tag{8}$$

where $\mathbf{v}$ is the velocity of the charge q_e or q_m, respectively. If a magnetic charge is placed in fields due to electric charges, it may possibly be influenced by a scalar magnetic field or a 1-form electric field. The only way to combine the magnetic charge q_m and velocity $\mathbf{v}$ with the fields E, D, H, B to produce a force 1-form is $q_m E$ or $q_m \mathbf{v} \lrcorner B$. Both of these terms are present in the first of the Lorentz force laws. If these interactions were found by experiment to give the correct force law, we would be forced to conclude that the test charge was identical with an electric charge. A similar argument holds for an electric test charge in a system of magnetic charges.

4. WAVE PROPAGATION

We now derive the wave equation for the electric charge case in 2-space. Applying $d\star$ to both sides of Faraday's law,

$$\begin{aligned} d\star dE &= -\frac{\partial}{\partial t} d\star B \\ &= -\mu \frac{\partial}{\partial t} dH. \end{aligned}$$

Substituting for H using Ampere's law with $J = 0$,

$$d\star dE = -\mu \frac{\partial^2}{\partial t^2} D$$

Starring both sides and using the constitutive relation for D,

$$\star d\star dE = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} E$$

The wave operator Δ satisfies the following identity,

$$\Delta = -\star d\star d + d\star d\star .$$

Using this identity,

$$-\Delta E + d\star d\star E = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} E$$

But $d\star d\star E = d\star d\frac{1}{\epsilon}D = d\star\frac{1}{\epsilon}\rho$. If there are no sources, $\rho = 0$, and we have the wave equation,

$$\Delta E = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E \tag{9}$$

which is in exactly the same form as the wave equation in three dimensions.

In rectangular coordinates, Eq. (9) becomes

$$\frac{\partial^2}{\partial x^2}E_1 + \frac{\partial^2}{\partial y^2}E_1 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E_1$$
$$\frac{\partial^2}{\partial x^2}E_2 + \frac{\partial^2}{\partial y^2}E_2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E_2$$

We know the solutions to an equation of this form. The simplest case is a time and spatially harmonic electric field,

$$E = Ee^{ikx}\,dy$$

where the time variation is suppressed and there is no component in the y direction. Hopefully, this will represent a plane wave. To be sure, we need to compute the magnetic field and see if it is propagating along with the electric field. Using Faraday's law,

$$\begin{aligned} B &= \frac{1}{i\omega}dE \\ &= \frac{k}{\omega}Ee^{ikx}\,dx\,dy. \end{aligned}$$

This tells us that the magnetic field intensity goes up and down with the same time and spatially harmonic variation as the electric field. While the electric field vector is moving back and forth in the 2-space, the scalar magnetic field is becoming alternately positive and negative.

5. TWO-DIMENSIONAL SPACE AS A SLICE OF THREE-SPACE

Consider a configuration of electrically charged sources in three-dimensional space with translational symmetry in the z direction. All charges must be line charges and by symmetry all fields must be independent of z.

By the symmetry of the sources, E and D can have no component in the z direction, and can be written

$$\begin{aligned} D &= D_1\,dy\,dz + D_2\,dz\,dx \\ E &= E_1\,dx + E_2\,dy \end{aligned}$$

where the coefficients are functions of x and y only. Ampere's law $dH = \frac{\partial}{\partial t}D + J$ becomes

$$\left(\frac{\partial}{\partial x}H_2 - \frac{\partial}{\partial y}H_1\right)dx\,dy - \frac{\partial}{\partial x}H_3\,dz\,dx + \frac{\partial}{\partial y}H_3\,dy\,dz = \frac{\partial}{\partial t}D_1\,dy\,dz + \frac{\partial}{\partial t}D_2\,dz\,dx + J_1\,dy\,dz + J_2\,dz\,dx \quad (10)$$

which shows that the x and y components of H must vanish. Maxwell's laws for sources of the given symmetry then become,

$$
\begin{aligned}
d(E_1\,dx + E_2\,dy) &= -\frac{\partial}{\partial t} B_z\,dx\,dy \\
dH_z\,dz &= \frac{\partial}{\partial t}(D_1\,dy\,dz + D_2\,dz\,dx) + J_1\,dy\,dz + J_2\,dz\,dx) \\
d(D_1\,dy\,dz + D_2\,dz\,dx) &= 0 \\
d(B_z\,dy\,dz) &= \rho \\
D_1\,dy\,dz + D_2\,dz\,dx &= \star(E_1\,dx + E_2\,dy) \\
B_z\,dx\,dy &= \star H_z\,dz
\end{aligned}
$$

The d operator can be rewritten as $(\frac{\partial}{\partial x}\,dx + \frac{\partial}{\partial y}\,dy)\wedge$ since the field components are independent of z. The factor of dz can then be removed from Faraday's law. dz can also be removed from the constitutive relations if the star operator becomes the two dimensional star operator given above. These equations then reduce to those found in Sec. 2.

6. ELECTRODYNAMICS IN A CURVED TWO-DIMENSIONAL SPACE

It is easy to write Maxwell's equations for the case of a curved two-dimensional space. In fact, the equations remain identical in form to those in Eq. 6. Only the star operator changes, since the star operator for a space depends on the metric of space, and the metric defines the curvature of the space. The behavior of the fields in a curved space is idential to the behavior of fields in an anisotropic medium where the permitivity and permeability tensors are both proportional to the metric of the curved space.

The question is, to what three-dimensional, *isotropic* situation does the curved, two-dimensional case correspond, in the manner of Sec. 5?

7. CONCLUSION

We have seen that it is easy to move from three- to two-dimensional electromagnetics using differential forms and the exterior derivative. The vector calculus can also be used to go the same route, but by contrast the derivation is not as clear and obvious as that above.