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Display and Computation of Winds in Oceanography and Meteorology

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Abstract

This report describes standard plotting and display conventions for winds and wind-derived fields used in ocean oceanography and meteorology. These are used to document plotting and computational algorithms for model-based wind field estimation. Emphasis is placed on computing the wind vorticity and divergence using mixed coordinate systems. Various examples using Seasat and ERS-1 scatterometer data are presented.

1 Introduction: Conventions for Describing Winds

Plotting and displaying wind fields are a common need when studying wind fields and their derivatives. Unfortunately much confusion arises due to the fact that there are several standard ways or coventions of plotting wind vector fields. This problem is complicated by the mixed coordinate systems which occur in scatterometry. Part of the purpose of this report is to clearly state these conventions and methods for converting from one to another and to treat display and plotting the the presence of mixed coordinate systems.

Separate conventions are used in meteorology and oceanography, with two frequently used conventions in oceanography. A common convention is accepted for the *component* (u, v) winds.

1.1 Meteorological Convention

Flags (barbs) are most commonly used to denote the wind direction and wind speed. However, vectors are also frequently used. When winds are plotted using vectors the direction of the arrow points *toward* the direction from which the wind flows, i.e., the arrow acts like an anemometer. Directions are measured clockwise (CW) from North. An angle of $0°$ denotes a *Northerly* wind flowing *from* North *to* South. An angle of $270°$ denotes a *Westerly* wind flowing *from* West *to* East. In the meteorological convention, the head of the arrow (i.e., the point) is displayed at the location of the measurement site. The tail extends outward from the measurement location.

1.2 Oceanographic Conventions

Unfortunately, two wind conventions for wind directions are used in oceanography. In both conventions, the arrow used to indicate the wind (or ocean) denotes the direction of mass flow. In the most common convention, wind angles are clockwise (CW) from North. A wind angle of $0°$ denotes a wind flowing *from* the South *to* the North and $90°$ denotes a wind flowing *from* West *to* East. Less commonly, the angle is counter-clockwise (CCW) relative to East. In both conventions the base or tail of the arrow is located at the measurement site with the head extending away from the measurement site.

1.3 Component Convention

Fortunately, the convention for the wind vector components is common to both arenas. A given wind vector $\vec{U}$ may be decomposed into two orthogonal *flow* components, u and v, i.e., $\vec{U} = (u, v)$. u is known as the *zonal* wind and is positive toward the East. v is the *meridonal* wind and is positive to the North. Thus, u and v make up a standard right-handed x, y coordinate systems. In this system the wind direction is counter-clockwise (CCW) from East so that a 0° wind denotes a zonal flow *toward* the East while a 90° wind denotes flow *toward* the North.

1.4 Hemispherical Flow

Given a weather map in which the convention is initially uncertain, the convention in use can be determined by looking at the wind flow around sharp lows. In the Nothern hemisphere, such a flow is CCW while in the Southern hemisphere it is CW.

1.5 Conversion Between Conventions

The meteorological convention and the primary oceanographic convention (relative to North) differ by 180°. Derivatives in one consistent coordinate system will be the negative of consistent derivatives in the other system. For the oceanographic convention, given a wind speed S and a wind direction relative to North ϕ_o, the u and v component winds are computed as

$$\begin{aligned} \phi_u &= 90^\circ - \phi_o && (1) \\ u &= S \cos \phi_u && (2) \\ v &= S \sin \phi_u. && (3) \end{aligned}$$

For the meteorological convention with wind direction ϕ_m

$$\begin{aligned} \phi_u &= 90^\circ - (\phi_m - 180^\circ) \\ &= -\phi_m - 90^\circ && (4) \\ u &= -S \cos \phi_u && (5) \\ v &= -S \sin \phi_u. && (6) \end{aligned}$$

To convert u and v component winds into oceanographic or meteorological convention wind speed and direction,

$$\begin{aligned} \phi_u &= \tan^{-1} v/u \quad \text{(four quadrant inverse)} && (7) \\ S &= \sqrt{u^2 + v^2} && (8) \\ \phi_o &= 90^\circ - \phi_u && (9) \\ \phi_m &= -\phi_u - 90^\circ. && (10) \end{aligned}$$

Note that none of these conversions are simple rotations.

2 Wind Coordinate Systems

Consider a two-dimension vector field $\vec{V}(x, y)$ with scalar component fields $V_x(x, y)$ and $V_y(x, y)$, i.e.,

$$\vec{V}(x, y) = V_x(x, y)\hat{x} + V_y(x, y)\hat{y}$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the x and y directions respectively.

In this report, x will be aligned East-West (positive toward East) and y will be aligned North-South (positive toward North) or $(x, y) = (E, N)$. Thus, $\hat{u} = \hat{x}$, $\hat{v} = \hat{y}$, $V_x(x, y) \stackrel{\triangle}{=} u(x, y)$ and $V_y(x, y) \stackrel{\triangle}{=} v(x, y)$ where the circumflex denotes a unit vector.

Our measurements of $\vec{V}(x, y)$ are made over a swath aligned with a rotated coordinate system (x', y') as shown in Fig. 1 for both ascending and descending orbits. The measurement system measures $V_x(x, y)$ and $V_y(x, y)$. It is also capable of providing estimates of the partial derivatives; however, it can only estimate derivatives along the (c, a) axis system, i.e., along (x', y'). The angle (always positive) between x and x' is $\alpha = \psi_g$. Note that for an ascending orbit x' corresponds to c and y' corresponds to a while for a desceding orbit x' corresponds to $-c$ and y' corresponds to $-a$ where the c, a axes indicate the cross-track/along-track data indexing.

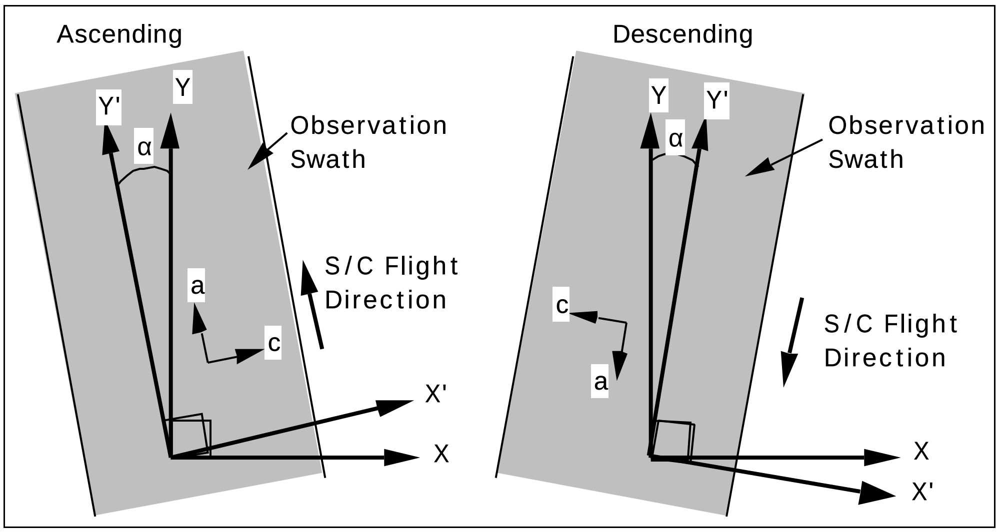

Figure 1: Rotational geometry

To express the component fields in the primed system as a function of the unprimed system using a simple rotational transformation as,

$$V_{x'}(x', y') = V_x(x, y) \cos\alpha + V_y(x, y) \sin\alpha \quad (11)$$

$$V_{y'}(x', y') = -V_x(x, y) \sin\alpha + V_y(x, y) \cos\alpha. \quad (12)$$

Similarly,

$$V_x(x, y) = V_{x'}(x', y') \cos\alpha - V_{y'}(x', y') \sin\alpha \quad (13)$$

$$V_y(x, y) = V_{x'}(x', y') \sin\alpha + V_{y'}(x', y') \cos\alpha. \quad (14)$$

These can be expressed in matrix form by first writting

$$\vec{V} = \begin{bmatrix} V_x(x, y) \\ V_y(x, y) \end{bmatrix} \quad (15)$$

and

$$\vec{V}' = \begin{bmatrix} V_{x'}(x', y') \\ V_{y'}(x', y') \end{bmatrix} \quad (16)$$

Then,

$$\vec{V}' = T(\alpha)\vec{V} \quad (17)$$
$$\vec{V} = T(-\alpha)\vec{V}' \quad (18)$$

where the 2×2 rotation matrix $T(\alpha)$ is defined as

$$T(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}. \quad (19)$$

To relate the primed (x', y') system to the cross/along-track (c, a) system,

$$V_{x'}(x', y') = \delta V_c(c, a) \quad (20)$$
$$V_{y'}(x', y') = \delta V_a(c, a) \quad (21)$$

where

$$\delta = \begin{cases} 1 & \text{ascending orbit} \\ -1 & \text{descending orbit.} \end{cases} \quad (22)$$

In matrix form this is

$$\vec{V}^\dagger = T(\delta' 180^\circ)\vec{V} \quad (23)$$
$$\vec{V} = T(\delta' 180^\circ)\vec{V}^\dagger \quad (24)$$

where

$$\vec{V}^\dagger = \begin{bmatrix} V_c(c, a) \\ V_a(c, a) \end{bmatrix} \quad (25)$$

and

$$\delta' = \begin{cases} 0 & \text{ascending orbit} \\ 1 & \text{descending orbit.} \end{cases} \quad (26)$$

2.1 Mixed Coordinates

Suppose,that we begin with $\vec{V}_o = [u_o, v_o]^t$ where

$$u_o = S\cos\phi_o \quad (27)$$
$$v_o = S\sin\phi_o. \quad (28)$$

where S is the wind speed and ϕ_o is the oceanographic convention wind direction. What is the relationship between $\vec{V}_o$ and $\vec{V} = [u, v]^t$?

Note that $\phi_u = 90^\circ - \phi_o$ and

$$u = S\cos\phi_u \quad (29)$$
$$v = S\sin\phi_u. \quad (30)$$

We see that,

$$\begin{aligned} \cos\phi_u &= \cos(90^\circ - \phi_o) \\ &= \cos(90^\circ)\cos\phi_o + \sin(90^\circ)\sin\phi_o \\ &= \sin\phi_o \end{aligned} \quad (31)$$

$$\begin{aligned} \sin\phi_u &= \sin(90^\circ - \phi_o) \\ &= \sin(90^\circ)\cos\phi_o - \cos(90^\circ)\sin\phi_o \\ &= \cos\phi_o. \end{aligned} \quad (32)$$

It follows that the u and v components of the wind expressed in conventional coordinates are a transformation of the u_o and v_o components, i.e., $u = v_o$ and $v = u_o$,

$$\vec{V} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_o \\ v_o \end{bmatrix} = R\vec{V}_o = -R\vec{V}_m. \tag{33}$$

where $\vec{V}_m = [S\cos\phi_m, S\sin\phi_m]^t = -\vec{V}_o$. However, the transformation defined by R can not be expressed as a simple rotation, i.e., by $T(\alpha)$. Note that R is its own inverse, i.e., $R = R^{-1}$.

2.2 Consistent Coordinates

To develop a consistent coordinate frame for model-based wind field estimation let us define the following:

$$\phi_{\text{mbe}} = \phi_u + \delta' 180^\circ = 90^\circ - \phi_o + \delta' 180^\circ = -90^\circ - \phi_m - \delta' 180^\circ \tag{34}$$

and

$$\phi_c = \phi_u + \delta' 180^\circ - \delta\alpha = 90^\circ - \phi_o + \delta' 180^\circ - \delta\alpha = -90^\circ - \phi_m + \delta' 180^\circ - \delta\alpha. \tag{35}$$

Then, ϕ_{mbe} gives the wind direction assuming that the along-track direction of scatterometer sampling swath (grid) is aligned with North-South and where the cross-track is aligned with East-West. This angle will be used in place of ϕ_c whenever α is unavailable.

The angle ϕ_c is the rotation of the wind vector to the scatterometer sampling grid. The quantity $u_c = S\cos\phi_c$ is the projection of u onto the positive cross-track axis and $v_c = S\sin\phi_c$ is the projection of v onto the along-track axis. Let $\vec{V}_c = [u_c, v_c]^t$ and $\vec{V} = [u, v]^t$. We see that,

$$\begin{aligned}
\cos\phi_c &= \cos(\phi_u + \delta' 180^\circ - \delta\alpha) \\
&= \cos\phi_u \cos(\delta' 180^\circ - \delta\alpha) - \sin\phi_u \sin(\delta' 180^\circ - \delta\alpha) \\
&= \cos\phi_u \left(\cos\delta' 180^\circ \cos\delta\alpha + \sin\delta' 180^\circ \sin\delta\alpha\right) - \sin\phi_u \left(\sin\delta' 180^\circ \cos\delta\alpha - \cos\delta' 180^\circ \sin\delta\alpha\right) \\
&= \cos\phi_u \delta \cos\delta\alpha + \sin\phi_u \delta \sin\delta\alpha \\
&= \delta\left(\cos\phi_u \cos\delta\alpha + \sin\phi_u \sin\delta\alpha\right) && (36) \\
&= \delta\left(\sin\phi_o \cos\delta\alpha + \cos\phi_o \sin\delta\alpha\right) && (37) \\
\sin\phi_c &= \sin(\phi_u + \delta' 180^\circ - \delta\alpha) \\
&= \sin\phi_u \cos(\delta' 180^\circ - \delta\alpha) + \cos\phi_u \sin(\delta' 180^\circ - \delta\alpha) \\
&= \sin\phi_u \left(\cos\delta' 180^\circ \cos\delta\alpha + \sin\delta' 180^\circ \sin\delta\alpha\right) + \cos\phi_u \left(\sin\delta' 180^\circ \cos\delta\alpha - \cos\delta' 180^\circ \sin\delta\alpha\right) \\
&= \sin\phi_u \delta \cos\delta\alpha - \cos\phi_u \delta \sin\delta\alpha \\
&= \delta\left(\sin\phi_u \cos\delta\alpha - \cos\phi_u \sin\delta\alpha\right) && (38) \\
&= \delta\left(\cos\phi_o \cos\delta\alpha - \sin\phi_o \sin\delta\alpha\right) && (39)
\end{aligned}$$

It follows that

$$\begin{aligned}
\vec{V}_c &= \delta T(+\delta\alpha)\vec{V} = \delta T(+\delta\alpha)R\vec{V}_o = -\delta T(+\delta\alpha)R\vec{V}_m && (40) \\
\vec{V} &= \delta T(-\delta\alpha)\vec{V}_c && (41) \\
\vec{V}_o &= \delta R T(-\delta\alpha)\vec{V}_c = -\vec{V}_m. && (42)
\end{aligned}$$

Thus, ϕ_c is a simple rotation (and sign change for descending orbits) of the u, v component winds. As will be shown below, this implies that the curl and divergence of $\vec{V}_c$ in the (c,a) system will be the same as the curl and divergence of $\vec{V}$ in the (u,v) [i.e., (x,y)=(E,N)] coordinate system. (Note, however, that the signs of the curl and divergence of $\vec{V}_c$ must be reversed, or multiplied by δ, for a descending orbit.) Therefore, ϕ_c provides a consistent coordinate transform between (c,a) and (x,y).

3 Scatterometer Dataset Winds and Wind Retrieval

This section documents the conventions uses in processed winds, the geophysical model function relating radar backscatter to winds, and the definitions of the relative the azimuth angle.

3.1 Wind Dataset conventions

Attached are plots for both ascending and descending passes using relative to North and MBE grid-relative plots for SASS and ERS-1 winds. The AES/Woiceshyn SASS winds are in the meteorological convention. Atlas/GSFC SASS wind directions are given as clockwise from North. I believe these winds are also given in the meteorlogical convention. ERS-1 scatterometer data (retrieved winds) is delivered from JPL (1993) in wind speed and direction. While the first data delieveries where identified as oceanographic convention, this was later corrected to be the meteorological convention with wind directions given with respect to North. Winds blowing from the West to the East have a direction of 90°. NSCAT has chosen to represent all winds in the oceanographic convention.

3.2 Wind Retrieval and the Geophysical Model Function

The geophysical model function (e.g., SASS1 or Wentz) relating wind to radar backscatter is indexed by the relative azimuth angle between the radar illumination and the wind. The key to the convention used is the definition of "upwind." The typical convention for defining upwind is that a 0° wind is "in your face," i.e., upwind corresponds to mass flow toward the observation site [R. Scott Dunbar, personal communication]. If upwind is aligned with North, this is the meteorological convention.

In the model function, the relative angle χ between the wind and the radar illumination is computed as

$$\chi = \psi_a - \phi \tag{43}$$

where ϕ is the wind direction (ϕ_m) and ψ_a is the angle of the radar illumination relative to North. Recently, the JPL point-wise wind retrieval algorithm was discovered to contain an inconsistency in the usage of the relative radar azimuth angle: winds retrieved assuming the meteorological convention for the beam azimuth angle relative to North are produced with the directions in the oceanographic convenction (i.e., the direction is reversed by 180°) [R. Scott Dunbar, personal communication].

Note that in the tabular model function, σ^o is assumed to be a symmetric function of χ, i.e., $\sigma^o(\chi) = \sigma^o(-\chi)$. Thus, the model function is only stored for $0° \leq \chi \leq 180°$. To look up σ^o is the table χ is computed as positive, modulo 360°. Then, if $\chi > 180°$, χ is recomputed as $360° - \chi$.

3.3 Relative Radar Azimuth Angle

Typically, the angle between the scatterometer flight direction and the illumination pattern is given as clock-wise from the flight direction. For SASS this gives azimuth angles of approximately $\psi = 45°$ and $\psi = 135°$ for the right-side swath. Given the direction (ascending or descending) of the orbit, the relative angle ψ_a between the radar illumination (which corresonds to antenna illumination) and North is computed. This is used to compute the relative azimuth angle between the wind and the radar illumination during the wind retrieval process.

NSCAT defines the relative beam azimuth as degrees CW from North so that for an ascending orbit the relative azimuth angle for the forward beam on the right side (looking from above in the direction of the flight) is just less than 45°. ERS-1, on the other hand, defines the relative angle differently. To compute the ERS-1 relative azimuth angle stand on the measurement cell facing North. Rotate CW so that you are facing back along the radar illumination to the spacecraft. Thus, for the right-side forward beam the azimuth angle is somewhat less than $45 + 180° = 225$ (i.e., $\psi_a = \psi - \alpha + 180°$) for an ascending swath. So, for a wind from North to South with the antenna at a relative azimuth angle of 0° to North (the spacecraft would be exactly North of the measurement cell), the relative radar illumination angle is 0° [R. Scott Dunbar, personal communication]. (See Fig. 2.)

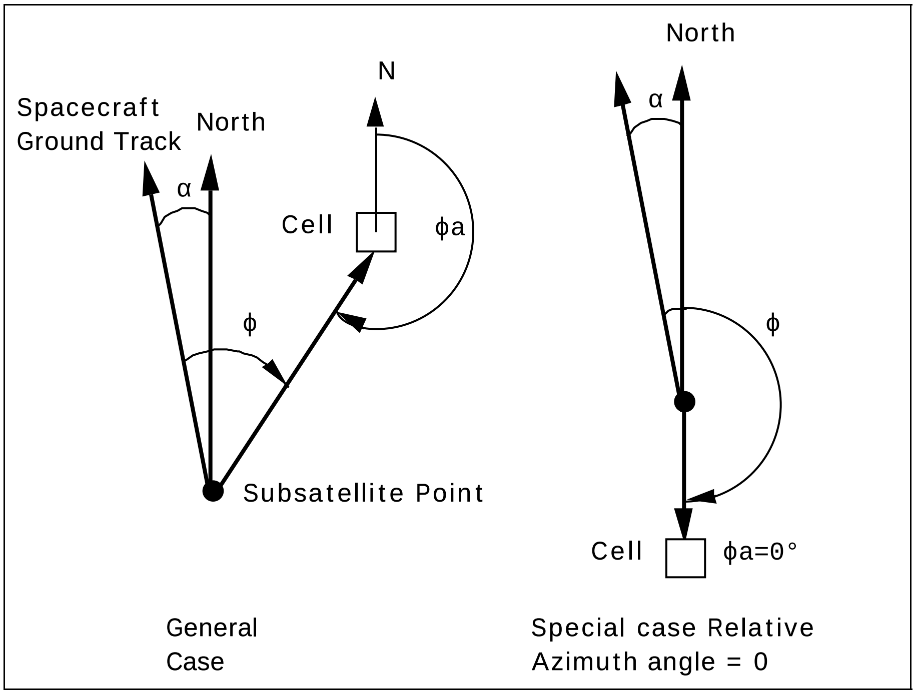

Figure 2: ERS-1 Relative Azimuth Geometry Examples

4 Vorticity and Divergence

The divergence of $\vec{V}(x,y)$ is defined as,

$$\begin{aligned} \text{Div}\{\vec{V}(x,y)\} &\triangleq \frac{\partial}{\partial x}V_x(x,y) + \frac{\partial}{\partial y}V_y(x,y) \\ &= \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v. \end{aligned} \tag{44}$$

The vorticity of $\vec{V}(x,y)$ is defined as,

$$\begin{aligned} \text{Vor}\{\vec{V}(x,y)\} &\triangleq -\frac{\partial}{\partial y}V_x(x,y) + \frac{\partial}{\partial x}V_y(x,y) \\ &= -\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v. \end{aligned} \tag{45}$$

4.1 Taking Partial Derivatives in Mixed Coordinate Systems

Given that the scatterometer samples the wind field in a uniform (x', y') grid and we are interested in wind components in the $(x,y) = (E,N)$ coordinate system, several questions arise: 1) Can we compute the partials with respect to the unprimed coordinate system, i.e. with respect to (x,y), from the primed coordinate system? 2) What is the relationship of the curl and diverence computed in the primed system with the unprimed system?

We note that the partials can be written as,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x} \tag{46}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}\frac{\partial y'}{\partial y} + \frac{\partial}{\partial x'}\frac{\partial x'}{\partial y}. \tag{47}$$

Since

$$\frac{\partial x'}{\partial x} = \cos\alpha \tag{48}$$

$$\frac{\partial y'}{\partial y} = \cos\alpha \tag{49}$$

$$\frac{\partial y'}{\partial x} = -\sin\alpha \tag{50}$$

$$\frac{\partial x'}{\partial y} = \sin\alpha, \tag{51}$$

it follows that

$$\frac{\partial}{\partial x} = \cos\alpha\frac{\partial}{\partial x'} - \sin\alpha\frac{\partial}{\partial y'} \tag{52}$$

$$\frac{\partial}{\partial y} = \cos\alpha\frac{\partial}{\partial y'} + \sin\alpha\frac{\partial}{\partial x'}. \tag{53}$$

Defining

$$\vec{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \tag{54}$$

and

$$\vec{\partial}' = \begin{bmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{bmatrix} \tag{55}$$

we see that this is a simple rotation,

$$\vec{\partial} = T(-\alpha)\vec{\partial}'. \tag{56}$$

We can reverse the process by noting that,

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\frac{\partial y}{\partial x'} \tag{57}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial}{\partial x}\frac{\partial x}{\partial y'}. \tag{58}$$

and

$$\frac{\partial x}{\partial x'} = \cos\alpha \tag{59}$$

$$\frac{\partial y}{\partial y'} = \cos\alpha \tag{60}$$

$$\frac{\partial y}{\partial x'} = \sin\alpha \tag{61}$$

$$\frac{\partial x}{\partial y'} = -\sin\alpha, \tag{62}$$

it follows that

$$\frac{\partial}{\partial x'} = \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y} \tag{63}$$

$$\frac{\partial}{\partial y'} = \cos\alpha\frac{\partial}{\partial y} - \sin\alpha\frac{\partial}{\partial x}. \tag{64}$$

which, again, is a simple rotation,

$$\vec{\partial}' = T(\alpha)\vec{\partial}. \tag{65}$$

4.2 Divergence and Vorticity in Mixed Coordinate Systems

We now take the divergence of our vector field $\vec{V}$ (dropping the arguments for simplicity), substituting from previous results,

$$\begin{aligned}
\mathrm{Div}_{(x,y)}\{\vec{V}\} &= \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y & (66)\\
&= \frac{\partial}{\partial x}\left[V_{x'}\cos\alpha - V_{y'}\sin\alpha\right] + \frac{\partial}{\partial y}\left[V_{x'}\sin\alpha + V_{y'}\cos\alpha\right] & (67)\\
&= \cos\alpha\left[\frac{\partial}{\partial x}V_{x'} + \frac{\partial}{\partial y}V_{y'}\right] - \sin\alpha\left[-\frac{\partial}{\partial y}V_{x'} + \frac{\partial}{\partial x}V_{y'}\right] & (68)\\
&= \cos\alpha\mathrm{Div}_{(x,y)}\{\vec{V}'\} - \sin\alpha\mathrm{Vor}_{(x,y)}\{\vec{V}'\} & (69)
\end{aligned}$$

which is a mixed coordinate system result. Making further substitutions,

$$\begin{aligned}
\mathrm{Div}_{(x,y)}\{\vec{V}\} &= \cos\alpha\left[\frac{\partial}{\partial x}V_{x'} + \frac{\partial}{\partial y}V_{y'}\right] - \sin\alpha\left[-\frac{\partial}{\partial y}V_{x'} + \frac{\partial}{\partial x}V_{y'}\right] & (70)\\
&= \cos\alpha\left[\cos\alpha\frac{\partial}{\partial x'}V_{x'} - \sin\alpha\frac{\partial}{\partial y'}V_{x'} + \cos\alpha\frac{\partial}{\partial y'}V_{y'} + \sin\alpha\frac{\partial}{\partial x'}V_{y'}\right] & (71)\\
&\quad - \sin\alpha\left[-\cos\alpha\frac{\partial}{\partial y'}V_{x'} - \sin\alpha\frac{\partial}{\partial x'}V_{x'}\cos\alpha\frac{\partial}{\partial x'}V_{y'} - \sin\alpha\frac{\partial}{\partial y'}V_{y'}\right] & (72)\\
&= \cos^2\alpha\left[\frac{\partial}{\partial x'}V_{x'} + \frac{\partial}{\partial y'}V_{y'}\right] + \sin^2\alpha\left[\frac{\partial}{\partial x'}V_{x'} + \frac{\partial}{\partial y'}V_{y'}\right] & (73)\\
&= \frac{\partial}{\partial x'}V_{x'} + \frac{\partial}{\partial y'}V_{y'} & (74)\\
&= \mathrm{Div}_{(x',y')}\{\vec{V}'\} & (75)
\end{aligned}$$

which shows that divergence is rotationally invariant.

Taking the vorticity of $\vec{V}$,

$$\mathrm{Vor}_{(x,y)}\{\vec{V}\} = -\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_y \tag{76}$$

$$
\begin{aligned}
&= -\frac{\partial}{\partial y}\left[V_{x'}\cos\alpha - V_{y'}\sin\alpha\right] + \frac{\partial}{\partial x}\left[V_{x'}\sin\alpha + V_{y'}\cos\alpha\right] && (77)\\
&= \cos\alpha\left[-\frac{\partial}{\partial y}V_{x'} + \frac{\partial}{\partial x}V_{y'}\right] + \sin\alpha\left[\frac{\partial}{\partial x}V_{x'} + \frac{\partial}{\partial y}V_{y'}\right] && (78)\\
&= \cos\alpha \mathrm{Vor}_{(x,y)}\{\vec{V}'\} + \sin\alpha \mathrm{Div}_{(x,y)}\{\vec{V}'\}, && (79)
\end{aligned}
$$

Which is the mixed coordinate system result. Making further substitutions,

$$
\begin{aligned}
\mathrm{Vor}_{(x,y)}\{\vec{V}\} &= \cos\alpha\left[-\frac{\partial}{\partial y}V_{x'} + \frac{\partial}{\partial x}V_{y'}\right] + \sin\alpha\left[\frac{\partial}{\partial x}V_{x'} + \frac{\partial}{\partial y}V_{y'}\right] && (80)\\
&= \cos\alpha\left[-\cos\alpha\frac{\partial}{\partial y'}V_{x'} - \sin\alpha\frac{\partial}{\partial x'}V_{x'} + \cos\alpha\frac{\partial}{\partial x'}V_{y'} - \sin\alpha\frac{\partial}{\partial y'}V_{y'}\right] && (81)\\
&\quad + \sin\alpha\left[\cos\alpha\frac{\partial}{\partial x'}V_{x'} - \sin\alpha\frac{\partial}{\partial y'}V_{x'} + \cos\alpha\frac{\partial}{\partial y'}V_{y'} + \sin\alpha\frac{\partial}{\partial x'}V_{y'}\right] && (82)\\
&= \cos^2\alpha\left[-\frac{\partial}{\partial y'}V_{x'} + \frac{\partial}{\partial x'}V_{y'}\right] + \sin^2\alpha\left[-\frac{\partial}{\partial y'}V_{x'} + \frac{\partial}{\partial x'}V_{y'}\right] && (83)\\
&= -\frac{\partial}{\partial y'}V_{x'} + \frac{\partial}{\partial y'}V_{x'} && (84)\\
&= \mathrm{Vor}_{(x',y')}\{\vec{V}'\} && (85)
\end{aligned}
$$

which shows that vorticity is rotationally invariant.

Defining,

$$
\begin{aligned}
Q_{(x,y)} &= \begin{bmatrix} \mathrm{Div}_{(x,y)}\{\vec{V}\} \\ \mathrm{Vor}_{(x,y)}\{\vec{V}\} \end{bmatrix} && (86)\\
Q'_{(x,y)} &= \begin{bmatrix} \mathrm{Div}_{(x,y)}\{\vec{V}'\} \\ \mathrm{Vor}_{(x,y)}\{\vec{V}'\} \end{bmatrix} && (87)\\
Q_{(x',y')} &= \begin{bmatrix} \mathrm{Div}_{(x',y')}\{\vec{V}\} \\ \mathrm{Vor}_{(x',y')}\{\vec{V}\} \end{bmatrix} && (88)\\
Q'_{(x',y')} &= \begin{bmatrix} \mathrm{Div}_{(x',y')}\{\vec{V}'\} \\ \mathrm{Vor}_{(x',y')}\{\vec{V}'\} \end{bmatrix} && (89)
\end{aligned}
$$

we can write

$$
Q_{(x,y)} = T(-\alpha)Q'_{(x,y)}. \qquad (90)
$$

Thus, the transformation of coordinates results in a rotation of the mixed coordinate divergence and vorticity fields. For the transformation in the other direction,

$$
\begin{aligned}
\mathrm{Div}_{(x',y')}\{\vec{V}'\} &= \frac{\partial}{\partial x'}V_{x'} + \frac{\partial}{\partial y'}V_{y'} && (91)\\
&= \frac{\partial}{\partial x'}\left[V_x\cos\alpha + V_y\sin\alpha\right] + \frac{\partial}{\partial y'}\left[-V_x\sin\alpha + V_y\cos\alpha\right] && (92)\\
&= \cos\alpha\left[\frac{\partial}{\partial x'}V_x + \frac{\partial}{\partial y'}V_y\right] + \sin\alpha\left[-\frac{\partial}{\partial y'}V_x + \frac{\partial}{\partial x'}V_y\right] && (93)\\
&= \cos\alpha \mathrm{Div}_{(x',y')}\{\vec{V}\} + \sin\alpha \mathrm{Vor}_{(x',y')}\{\vec{V}\} && (94)
\end{aligned}
$$

and

$$\text{Vor}_{(x',y')}\{\vec{V}'\} = -\frac{\partial}{\partial y'}V_{x'} + \frac{\partial}{\partial x'}V_{y'} \tag{95}$$

$$= -\frac{\partial}{\partial y'}\left[V_x \cos\alpha + V_y \sin\alpha\right] + \frac{\partial}{\partial x'}\left[-V_x \sin\alpha + V_y \cos\alpha\right] \tag{96}$$

$$= \cos\alpha\left[-\frac{\partial}{\partial y}V_x + \frac{\partial}{\partial x}V_{y'}\right] - \sin\alpha\left[\frac{\partial}{\partial x}V_{x'} + \frac{\partial}{\partial y}V_y\right] \tag{97}$$

$$= \cos\alpha \text{Vor}_{(x',y')}\{\vec{V}\} - \sin\alpha \text{Div}_{(x',y')}\{\vec{V}\} \tag{98}$$

so that

$$Q'_{(x',y')} = T(\alpha)Q_{(x',y')}. \tag{99}$$

Since the vorticity and divergence fields are rotationally invariant (i.e., their values are not dependent on the coordinate system chosen) we have,

$$Q_{(x,y)} = Q'_{(x',y')} \tag{100}$$

$$T(-\alpha)Q'_{(x,y)} = T(\alpha)Q_{(x',y')}. \tag{101}$$

A relationship between the mixed coordinate results can be obtained, i.e.,

$$Q'_{(x,y)} = T^{-1}(-\alpha)T(\alpha)Q_{(x',y')} \tag{102}$$

$$Q'_{(x,y)} = T(\alpha)T(\alpha)Q_{(x',y')} \tag{103}$$

$$Q'_{(x,y)} = T^2(\alpha)Q_{(x',y')} \tag{104}$$

$$Q'_{(x,y)} = \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix} Q_{(x',y')}. \tag{105}$$

It follows that

$$\text{Div}_{(x,y)}\{\vec{V}'\} = \beta \text{Div}_{(x',y')}\{\vec{V}\} \tag{106}$$

$$\text{Vor}_{(x,y)}\{\vec{V}'\} = \beta \text{Vor}_{(x',y')}\{\vec{V}\} \tag{107}$$

where

$$\beta = \cos^2\alpha - \sin^2\alpha = \cos 2\alpha. \tag{108}$$

5 Estimating the Vorticity and Divergence

Consider a wind field $\vec{V_o}$ specified by a scalar speed $S(c_i, a_j)$ and direction $\phi_o(c_i, a_j)$ fields sampled on rectangular grid in the c, a coordinate system where $c_i = i\Delta c$ and $a_j = j\Delta a$. $\phi_o(c_i, a_j)$ is the oceanographic convention angle relative to true North. Then,

$$u(c_i, a_j) = S(c_i, a_j)\cos\phi_u(c_i, a_j) \tag{109}$$

$$v(c_i, a_j) = S(c_i, a_j)\sin\phi_u(c_i, a_j) \tag{110}$$

where $\phi_u(c_i, a_j) = 90^\circ - \phi_o(c_i, a_j)$.

A first-order estimate of the partials of u and v with respect to c and a are,

$$\begin{aligned}
\frac{\partial}{\partial c}u(c,a) &\approx \frac{1}{\Delta c}\left[u(c_i,a_j) - u(c_{(i-1)},a_j)\right] && (111)\\
\frac{\partial}{\partial c}v(c,a) &\approx \frac{1}{\Delta c}\left[v(c_i,a_j) - v(c_{(i-1)},a_j)\right] && (112)\\
\frac{\partial}{\partial a}u(c,a) &\approx \frac{1}{\Delta a}\left[u(c_i,a_j) - u(c_i,a_{(j-1)})\right] && (113)\\
\frac{\partial}{\partial a}v(c,a) &\approx \frac{1}{\Delta a}\left[v(c_i,a_j) - v(c_i,a_{(j-1)})\right]. && (114)
\end{aligned}$$

Noting that y' is in the same direction of a for ascending orbits we obtain

$$\begin{aligned}
\frac{\partial}{\partial y'}u &\approx \frac{1}{\Delta a}\left[u(c_i,a_j) - u(c_i,a_{(j-1)})\right] && (115)\\
\frac{\partial}{\partial y'}v &\approx \frac{1}{\Delta a}\left[v(c_i,a_j) - v(c_i,a_{(j-1)})\right] && (116)
\end{aligned}$$

while for descending orbits we obtain

$$\begin{aligned}
\frac{\partial}{\partial y'}u &\approx -\frac{1}{\Delta a}\left[u(c_i,a_j) - u(c_i,a_{(j-1)})\right] && (117)\\
\frac{\partial}{\partial y'}v &\approx -\frac{1}{\Delta a}\left[v(c_i,a_j) - v(c_i,a_{(j-1)})\right]. && (118)
\end{aligned}$$

Since x' is in the the same direction as c we obtain the following for ascending orbits

$$\begin{aligned}
\frac{\partial}{\partial x'}u &\approx \frac{1}{\Delta c}\left[u(c_i,a_j) - u(c_{(i-1)},a_j)\right] && (119)\\
\frac{\partial}{\partial x'}v &\approx \frac{1}{\Delta c}\left[v(c_i,a_j) - v(c_{(i-1)},a_j)\right] && (120)
\end{aligned}$$

and since x' is in the opposite direction of c

$$\begin{aligned}
\frac{\partial}{\partial x'}u &\approx -\frac{1}{\Delta c}\left[u(c_i,a_j) - u(c_{(i-1)},a_j)\right] && (121)\\
\frac{\partial}{\partial x'}v &\approx -\frac{1}{\Delta c}\left[v(c_i,a_j) - v(c_{(i-1)},a_j)\right] && (122)
\end{aligned}$$

for descending orbits. Using the definition of δ in Eq. (22) these equations can be unified,

$$\begin{aligned}
\frac{\partial}{\partial y'}u &\approx \frac{\delta}{\Delta a}\left[u(c_i,a_j) - u(c_i,a_{(j-1)})\right] && (123)\\
\frac{\partial}{\partial y'}v &\approx \frac{\delta}{\Delta a}\left[v(c_i,a_j) - v(c_i,a_{(j-1)})\right] && (124)\\
\frac{\partial}{\partial x'}u &\approx \frac{\delta}{\Delta c}\left[u(c_i,a_j) - u(c_{(i-1)},a_j)\right] && (125)\\
\frac{\partial}{\partial x'}v &\approx \frac{\delta}{\Delta c}\left[v(c_i,a_j) - v(c_{(i-1)},a_j)\right]. && (126)
\end{aligned}$$

We can then apply the previous results to obtain the vorticity and divergence in the primed (x', y') system,

$$\begin{aligned}
\mathrm{Div}_{(x',y')}\{\vec{V}(x,y)\} &= \frac{\partial}{\partial x'}u + \frac{\partial}{\partial y'}v && (127)\\
\mathrm{Vor}\{\vec{V}(x,y)\} &= -\frac{\partial}{\partial y'}u + \frac{\partial}{\partial x'}v. && (128)
\end{aligned}$$

which is rotated to obtain the vorticity and divergence in the unprimed (x, y) system,

$$\begin{bmatrix} \text{Div}_{(x,y)}\{\vec{V}\} \\ \text{Vor}_{(x,y)}\{\vec{V}\} \end{bmatrix} = T(\alpha) \begin{bmatrix} \text{Div}_{(x',y')}\{\vec{V}\} \\ \text{Vor}_{(x',y')}\{\vec{V}\} \end{bmatrix}. \tag{129}$$

5.1 Mixed Coordinates

Suppose, however, that we begin with $\vec{V}_o = [u_o, v_o]^t$ where

$$u_o(c_i, a_j) = S(c_i, a_j) \cos \phi_o(c_i, a_j) \tag{130}$$

$$v_o(c_i, a_j) = S(c_i, a_j) \sin \phi_o(c_i, a_j) \tag{131}$$

and the partials

$$\frac{\partial}{\partial c} u_o(c, a) \approx \frac{1}{\Delta c} \left[u_o(c_i, a_j) - u_o(c_{(i-1)}, a_j) \right] \tag{132}$$

$$\frac{\partial}{\partial c} v_o(c, a) \approx \frac{1}{\Delta c} \left[v_o(c_i, a_j) - v_o(c_{(i-1)}, a_j) \right] \tag{133}$$

$$\frac{\partial}{\partial a} u_o(c, a) \approx \frac{1}{\Delta a} \left[u_o(c_i, a_j) - u_o(c_i, a_{(j-1)}) \right] \tag{134}$$

$$\frac{\partial}{\partial a} v_o(c, a) \approx \frac{1}{\Delta a} \left[v_o(c_i, a_j) - v_o(c_i, a_{(j-1)}) \right] \tag{135}$$

with the mixed coordinate divergence and curl

$$\text{Div}_{(c,a)}\{\vec{V}_o\} = \frac{\partial}{\partial c} u_o(c, a) + \frac{\partial}{\partial a} v_o(c, a) \tag{136}$$

$$\text{Vor}_{(c,a)}\{\vec{V}_o\} = -\frac{\partial}{\partial a} u_o(c, a) + \frac{\partial}{\partial c} v_o(c, a). \tag{137}$$

These are the equations which have been used to-date in the wind field model.

From the geometry of the problem we see that for an ascending orbit

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial c} \tag{138}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial a} \tag{139}$$

and for a descending orbit,

$$\frac{\partial}{\partial x'} = -\frac{\partial}{\partial c} \tag{140}$$

$$\frac{\partial}{\partial y'} = -\frac{\partial}{\partial a} \tag{141}$$

can be expressed as

$$\frac{\partial}{\partial x'} = \delta \frac{\partial}{\partial c} \tag{142}$$

$$\frac{\partial}{\partial y'} = \delta \frac{\partial}{\partial a}. \tag{143}$$

But, since $u = v_o$ and $v = u_o$,

$$\frac{\partial}{\partial x'} u_o = \frac{\partial}{\partial x'} v \tag{144}$$
$$\frac{\partial}{\partial x'} v_o = \frac{\partial}{\partial x'} u \tag{145}$$
$$\frac{\partial}{\partial y'} u_o = \frac{\partial}{\partial y'} v \tag{146}$$
$$\frac{\partial}{\partial y'} v_o = \frac{\partial}{\partial y'} u. \tag{147}$$

It then follows that

$$\frac{\partial}{\partial x'} u = \frac{\partial}{\partial x'} v_o = \delta \frac{\partial}{\partial c} v_o \approx \frac{\delta}{\Delta c} \left[v_o(c_i, a_j) - v_o(c_{(i-1)}, a_j) \right] \tag{148}$$
$$\frac{\partial}{\partial x'} v = \frac{\partial}{\partial x'} u_o = \delta \frac{\partial}{\partial c} u_o \approx \frac{\delta}{\Delta c} \left[u_o(c_i, a_j) - u_o(c_{(i-1)}, a_j) \right] \tag{149}$$
$$\frac{\partial}{\partial y'} u = \frac{\partial}{\partial y'} v_o = \delta \frac{\partial}{\partial a} v_o \approx \frac{\delta}{\Delta a} \left[v_o(c_i, a_j) - v_o(c_i, a_{(j-1)}) \right] \tag{150}$$
$$\frac{\partial}{\partial y'} v = \frac{\partial}{\partial y'} u_o = \delta \frac{\partial}{\partial a} u_o \approx \frac{\delta}{\Delta a} \left[u_o(c_i, a_j) - u_o(c_i, a_{(j-1)}) \right]. \tag{151}$$

With this result we can compute the divergence and vorticity of the (u, v) wind in the (x', y') coordinate system using Eqs. (127) and (128), apply the rotation given in Eq. (129), and determine the consistent divergence and vorticity of the (u, v) wind in the (x, y) coordinate system.

Examination of the mixed coordinate divergence and vorticity in Eq. (136) reveals,

$$\delta \mathrm{Div}_{(c,a)}\{\vec{V}_o\} = \delta \left(\frac{\partial}{\partial c} u_o(c, a) + \frac{\partial}{\partial a} v_o(c, a) \right) \tag{152}$$
$$= \frac{\partial}{\partial x'} u_o + \frac{\partial}{\partial y'} v_o \tag{153}$$
$$= \frac{\partial}{\partial x'} v + \frac{\partial}{\partial y'} u \tag{154}$$
$$\delta \mathrm{Vor}_{(c,a)}\{\vec{V}_o\} = \delta \left(-\frac{\partial}{\partial a} u_o(c, a) + \frac{\partial}{\partial c} v_o(c, a) \right) \tag{155}$$
$$= -\frac{\partial}{\partial y'} u_o + \frac{\partial}{\partial x'} v_o \tag{156}$$
$$= -\frac{\partial}{\partial y'} v + \frac{\partial}{\partial x'} u \tag{157}$$

As can be seen, there is no simple way to compute the consistent vorticity and divergence from this mixed vorticity and divergence.

6 Wind Field Plotting

Let us now consider the problem of plotting wind vectors for display. This requires transforming the wind vectors from their internal storage convention to a plotting coordinate system (x_p, y_p). The graphics display and hardcopy output uses a right-handed (x, y) coordinate system with x_p defined postive along the horizontal axis and y_p positive in the vertical direction. Angles are counter-clockwise (CCW) from the x_p

axis. To display wind vectors we use a subroutine which plots an arrow of specified length (proportional to the wind speed) with a tail at a specified location and the head at a given CCW angle ϕ_p from the x_p axis. By convention, the arrow will display the direction of the atmospheric *flow*. To enable the display of the wind direction when the wind speed is small, frequently a minimum length arrow (corresponding to 0 m/s) is used. To this is added a length proportional to the wind speed.

There are two primary display formats used in research at BYU: 1) a rectalinear grid in latitude and longitude (a lat/lon grid) and 2) a rectalinear grid in cross-track and along-track (the mbe grid). Additionally, there are two options for mbe grid plotting.

6.1 Lat/Lon Grid Format

For lat/lon grid plotting the longitude is aligned with x_p with positive x_p corresponding to East. Latitude is aligned with y_p with positive y_p corresponding to North. This results in aligning u (x) with x_p and v (y) with y_p. As a result

$$\phi_a = \phi_u = 90^\circ - \phi_o = \phi_c - \delta' 180^\circ. \tag{158}$$

This is conceptually the simplest plotting format. The tails of the arrows are located at the lat/lon locations specified for each wind vector. This is either computed from the σ^o measurement data set (the average location of the measurements) or by locating the centers of the wind vector cells on the cross-track/along-track grid and computing the spatial transformation from this coordinate system to lat/lon. A full one-half orbit swath will curve on in this plot format.

6.2 MBE Grid Formats

For plotting in the mbe grid format, we must worry about whether the orbit pass being plotted is ascending or descending. There are two primary options for mbe grid plotting: 1) plotting all directions with respect to North and 2) plotting all directions with respect to the cross-track/along-track grid. We note that option 1 can not be done if the angle of the grid with respect to North, α, is not known. (Note that due to the curvature of the Earth, α is slightly different for each grid element.)

Figure 3 illustrates the coordinate systems for each option. In both options the along-track coordinate is aligned with x_p with positive x_p corresponding to increasing along-track distance. Cross-track distance is aligned with y_p with *negative* y_p corresponding to increasing cross-track distance. Land is plotted from a map data set.

6.2.1 Plotting with Respect to MBE Grid

This is the simplest option for mbe grid plotting. In this format option, winds are plotted with respect to the mbe grid without regard to absolute north. All that is required is that vector directions be consistent with the (c,a) axes. To do this we let

$$\phi_p = \phi_c - 90^\circ = \phi_u - \delta 90^\circ = \phi_{\text{mbe}} - 90^\circ. \tag{159}$$

Land is indicated using land flags in the data set. Grid elements with no measurements may not have land flags even if the grid element is over land.

6.2.2 Plotting with Respect to North

For plotting with respect to North the angle α of the mbe grid with respect to North must be known. Then,

$$\phi_p = \phi_c - 90^\circ = \phi_u - \delta 90^\circ - \delta\alpha = \phi_{\text{mbe}} - 90^\circ - \delta\alpha. \tag{160}$$

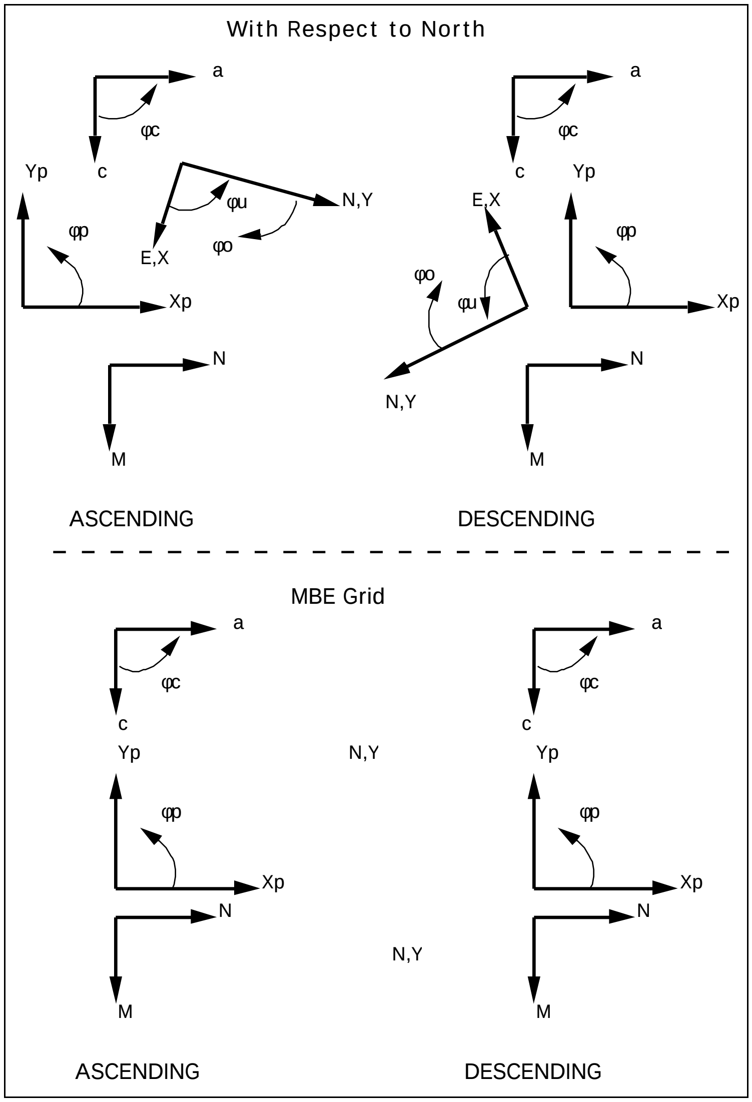

Figure 3: Plotting geometry

Tails of the wind vector arrows are located at the centers of the cross-track/along track grid elements. Land is indicated using land flags in the data set. Grid elements with no measurements may not have land flags even if the grid element is over land. A spatial transformation must be used to convert latitude and longitude to cross-track and along-track coordinates which may then be plotted. Latitude/longitude lines curve in this plot format.